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The spectral dynamics of laminar convection

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The non-linear equations of the Bénard convection problem are transformed to the spectral domain. The spectral basis consists of the supercritical normal modes of the characteristic-value problem in which the exponential growth rates are characteristic values. The norm of the spectrum is the variance of an arbitrary finite-amplitude state of convection. The equations that govern the spectrum are solvable by linear methods when the spectrum is truncated by exclusion of all convective modes except those of lowest-order symmetric vertical structure. Numerical computations of heat flux for a spectrum that contains only one convective mode are in good agreement with experimental data for water in the laminar régime.

1. Introduction

This investigation is concerned with free, vertical heat convection in a horizontal, thin layer of liquid—the Bénard problem. The state of knowledge of the linear stability theory of this problem is virtually complete (Chandrasekhar 1961), and establishes a base from which finite-amplitude, non-linear aspects of the problem can be investigated. Among the questions that lie in the latter field are determination of heat flux as a function of thermal potential, explanation of the horizontal plan form of convection cells, specification of the structure of thermal boundary layers, and prediction of transition from laminar to turbulent convection. (Some recent papers are cited at the end of this section.)

The aim of the present work is to develop a formal, spectral theory of the Bénard problem, and by this means, to make a contribution to the first of the questions noted above. In particular, a deductive theory of the heat flux is given for Rayleigh numbers of order 10^4 where at most a few convective modes are self-excited and convection is still laminar. The point of departure is the representation of a non-linear state, corresponding to a particular supercritical Rayleigh number, as a linear combination of the normal (linear) modes that correspond to the same Rayleigh number. To carry out this program it is necessary first to formulate the normal-mode theory in a general way that is convenient for a spectral representation; then to solve the normal-mode equations in order to make a quantitative determination of the structure of the most important supercritical modes.

Formulation of the normal-mode theory is along lines similar to those given, for example, in the recent book by Eckart (1960), but requires an extension of the latter analysis to include effects of viscosity and heat conduction. An important aspect of this formulation is that it proceeds directly from the momentum form of the hydrodynamical equations, rather than from any derived form obtained by elimination of variables. This leads to a normal-mode problem in which the exponential growth rates are characteristic values. The physical interpretation of the normal-mode spectrum used in this study lies in the fact that the norm of the spectrum (the square sum of spectral amplitudes) is equal to the kinetic plus thermal 'variance' of the given state of finite-amplitude convection.

The foregoing procedure makes it possible to represent an arbitrary state of finite-amplitude convection, associated with particular values of the Rayleigh number and Prandtl number, as a linear combination of normal modes associated with the same values of these parameters. Formal spectral inversion of the governing hydrodynamical equations then leads to spectral-dynamic equations, which are a statement of the governing equations in the spectral domain. This process is familiar in turbulence theory, where the conversion usually is made by means of Fourier transforms. However, it should not be overlooked that the spectral domain may be well-suited for problems in laminar as well as turbulent motion. Moreover, formal development of the spectral theory is in most respects more straightforward in terms of the modes associated with the variance spectrum than by means of Fourier transforms. For the Bénard problem, the theory is in any case mathematically elementary, and for this reason is developed here from first principles, without any attempt to draw parallels with turbulence theory.

The spectral-dynamic equations can be regarded as posing an initial-value problem, the solution of which gives the evolution of the spectrum from an initially prescribed state (for example, see Saltzman 1962 and Herring 1963). Alternatively, if the time-derivative terms are put equal to zero (and if the spectrum is discrete), they are an infinite set of quadratic algebraic equations. If a solution of the latter equations exists for prescribed values of external parameters, it gives the equilibrium spectrum that corresponds to a steady (although not necessarily stable) state of convection. The work reported here is concerned only with this steady-state view of the problem. Moreover, only laminar convection is considered; thus, the spectrum is at equilibrium because the motion actually is steady, rather than as a result of stationary turbulence processes.

The spectral-dynamic equations cannot be solved exactly. However, in moderately supercritical states where the motion is still laminar and steady convection is possible, the primary self-excited mode may be expected to dominate the spectrum. It follows that good estimates in this source region of the spectrum may come from an approximation in which the spectrum is truncated by excluding all modes except the primary self-excited mode and those that predominate in interactions involving this mode. If truncation of the spectrum is severe, it may be possible to solve the correspondingly truncated spectral-dynamic equations without excessive numerical labour. This is the underlying approach that is used in the present investigation.

The programme outlined above represents in a sense an extension of an earlier investigation by Kuo & Platzman (1961, modified subsequently by Kuo 1961). However, because the former study was based upon marginal rather than supercritical states, the outcome was not completely satisfactory from the standpoint of spectral analysis. The basic idea of spectral analysis and truncation has been used in a variety of hydrodynamical problems, beginning with the attempt by Reynolds to arrive at a criterion for the instability of Poiseuille flow. An investigation that has particular relevance for the present work is that of Stuart (1958), who showed that estimates of Reynolds stress in slightly supercritical states can be obtained from quadratic integrals of the governing equations. The use of such integrals to derive estimates of the critical Reynolds number was demonstrated first by Reynolds himself. Stuart carries their application one step further, by showing that they provide a basis for estimating the amplitude of the secondary flow as well. In the examples given by Stuart the secondary flow is represented in terms of the marginal state of the self-excited mode, but the method also applies to a representation in terms of the spectrum associated with the supercritical state for which the amplitude of secondary flow is sought. Speaking now of the Bénard problem, it is shown in this paper that the heat flux obtained from Stuart's method applied in the latter sense is the same as that obtained from the spectral-dynamic equations by a truncation that excludes all convective modes except the primary self-excited mode, but includes in general all antisymmetric thermal modes (of which there is an infinite number). It should be added that this equivalence is contingent upon the use of the integral for kinetic plus thermal variance, rather than that for thermal variance alone.

Before proceeding further, I must take note of several recent papers which have points in common with methods or results to be given here. (As is inevitable in a field undergoing rapid cultivation, some of these were published or came to my attention after the present work was completed.) A paper by Spiegel (1962) is especially germane, since it utilizes the spectrum of supercritical normal modes, as well as the distinction between what I have called 'free' and 'forced' convective modes (see also Ledoux, Schwartzschild & Spiegel 1961). The impossibility of steady, finite-amplitude convection at subcritical Rayleigh number (§ 4) has been proved also by Howard (1963) and Sani (1964). The papers by Palm (1960), Segal & Stuart (1962) and Segel (1962) certainly are essential for an understanding of the stability properties of truncated spectral equations, and the related problem of the horizontal plan form of convection. Finally, reference should be made to the well-known mathematical similarity between the problem of heat convection across horizontal, parallel planes and that of momentum convection across rotating, concentric cylinders. The work of Stuart (1960), Watson (1960) and Davey (1962) is particularly noteworthy in connection with finite-amplitude aspects of the latter problem (see also Stuart 1963).

2. The governing equations

The equation of state is assumed to be $\rho' = \rho_0'(1-\alpha'\vartheta')$, where ρ_0' can be regarded as a mean density of the fluid as a whole, so that ϑ' is the excess of temperature over that which corresponds to mean density. Variations of the thermal expansion coefficient α', the thermal diffusivity κ' and the kinematic viscosity ν' are ignored.

The temperature field ϑ' can be split into a linear, static part $\vartheta_0' - R'z'$ plus a residual, dynamic part T'. Here $R' \equiv \Delta\vartheta'/d'$ is the uniform static-temperature 'lapse rate', where d' is the distance between horizontal boundaries and $\Delta\vartheta'$ is lower-boundary temperature minus upper-boundary temperature; thus, $\Delta\vartheta'$ and R' are *positive* for a statically *un*stable arrangement. The axis of z' is vertical and its direction is opposite to that of gravity.

For reduction to non-dimensional quantities we adopt the following units: length d'; time $d'^2(\kappa'\nu')^{-\frac{1}{2}}$; temperature $\kappa'\nu'(g'\alpha'd'^3)^{-1}$. The dimensionless viscosity coefficient, diffusivity coefficient and static-temperature gradient are then

$$\nu = (\nu'/\kappa')^{\frac{1}{2}} = P^{\frac{1}{2}},$$

$$\kappa = (\kappa'/\nu')^{\frac{1}{2}} = P^{-\frac{1}{2}},$$

$$R = R'g'\alpha'd'^4/\kappa'\nu',$$

where $P \equiv \nu'/\kappa'$ is the Prandtl number and R is the Rayleigh number. Note that $\nu\kappa = 1$, a relation used frequently in the sequel. Further, the dimensionless value of $g'\alpha'$ is $g\alpha = 1$.

The non-dimensional equations governing conservation of momentum, energy and mass are

$$\partial\mathbf{v}/\partial t = -\mathbf{v}\,.\,\nabla\mathbf{v} + \nu\nabla^2\mathbf{v} + \mathbf{k}T - \nabla\varpi, \tag{2.1a}$$

$$\partial T/\partial t = -\mathbf{v}\,.\,\nabla T + \kappa\nabla^2 T + R\mathbf{k}\,.\,\mathbf{v}, \tag{2.1b}$$

$$\nabla\,.\,\mathbf{v} = 0, \tag{2.2}$$

where $\mathbf{v}$ and ∇ are the three-dimensional velocity vector and gradient operator, and $\mathbf{k}$ is an upward-directed unit vector. In the first equation ϖ is the fluid pressure divided by the mean density, on the understanding that we exclude from ϖ the hydrostatic pressure associated with the mean density and with the static-density gradient that corresponds to $\vartheta_0 - Rz$, so that only the buoyancy term $\mathbf{k}T$ remains in the gravitational force. These equations incorporate the Boussinesq approximations, the principal feature of which is neglect of effects of dynamically-produced density variations upon momentum and mass fluctuations. Similar effects of static-density variations also are neglected here because we deal with a thin layer of fluid.†

Owing to the solenoidal character of $\mathbf{v}$, the pressure field ϖ is a passive variable in (2.1). In other words, for prescribed values of external parameters (R and P) and given boundary conditions, the state of convection can be regarded as completely specified by the vector

$$\mathbf{S} \equiv \begin{pmatrix} \mathbf{v} \\ T \end{pmatrix}, \tag{2.3}$$

since ϖ is determined implicitly by $\mathbf{v}, T$ through the equation that results by taking the divergence of (2.1a). The governing equation for the state vector $\mathbf{S}$ is

$$\partial\mathbf{S}/\partial t = -\mathbf{v}\,.\,\nabla\mathbf{S} + \mathscr{L}\mathbf{S} - \mathbf{G}, \tag{2.4}$$

$$\mathscr{L} \equiv \begin{pmatrix} \nu\nabla^2 & \mathbf{k} \\ R\mathbf{k}\,. & \kappa\nabla^2 \end{pmatrix}, \quad \mathbf{G} \equiv \begin{pmatrix} \nabla\varpi \\ 0 \end{pmatrix},$$

† Further details are given in the book by Chandrasekhar (1961).

an obvious restatement of (2.1). The linear operator $\mathscr{L}$ expresses the effects of viscosity ($\nu\nabla^2$), buoyancy (**k**), heat conduction ($\kappa\nabla^2$), and heat convection in the static part of the temperature field ($R\mathbf{k}.$). It is a function of R and P, and represents all linear physical processes relevant to the Bénard problem. Non-linear effects are contained in the convection term $\mathbf{v}.\nabla\mathbf{S}$, and implicitly in the pressure-gradient term **G**. As explained above, the latter is a function of **S**; however, the explicit form of this relation is not needed in the subsequent analysis.

A quantity that is fundamental for spectral analysis of (2.4) is the 'inner product' of two states of convection **S** and **S**$'$

$$\{\mathbf{S}, \mathbf{S}'\} \equiv \overline{\overline{\mathbf{v}.\mathbf{v}' + R^{-1}TT'}}. \tag{2.5}$$

Here $\overline{\overline{(\;)}}$ signifies a volume integration that extends over the entire space in which the fluid is contained, or over one complete cell if the motion is assumed to be spatially periodic. The two states **S** and **S**$'$ whose product is taken in the manner defined in (2.5) must be associated with the same external parameters R and P and with the same boundary conditions. This provision is necessary for self-adjointness of the operator $\mathscr{L}$ that appears in (2.4), as will be seen below.

An important special case of (2.5) is $\mathbf{S}' = \mathbf{S}$, which gives the *variance* of **S**

$$\{\mathbf{S}, \mathbf{S}\} = \overline{\overline{\mathbf{v}^2 + R^{-1}T^2}}. \tag{2.6}$$

Inasmuch as this investigation is concerned exclusively with $R > 0$ (gravitational *in*stability), the product $\{\mathbf{S}, \mathbf{S}\}$ is positive definite for any **S**.

Balance equations for $K \equiv \mathbf{v}^2$ and $H \equiv R^{-1}T^2$ come from multiplication of (2.1*a*) by $2\mathbf{v}$ and (2.1*b*) by $2R^{-1}T$. After transformation with the aid of (2.2), the balance equations are

$$\partial K/\partial t = 2wT - \operatorname{div}\mathbf{f}_K - 4\nu e_{ij}e_{ij}, \tag{2.7a}$$

$$\partial H/\partial t = 2wT - \operatorname{div}\mathbf{f}_H - 2\nu R^{-1}(\nabla T)^2. \tag{2.7b}$$

They show that there is equipartition in production of kinetic and thermal variance, at the rate $2wT$ (proportional to upward convective heat flux). The fluxes $\mathbf{f}_K \equiv \mathbf{v}K - 4\nu\mathbf{v}.\mathscr{E} + 2\mathbf{v}\varpi$ and $\mathbf{f}_H \equiv \mathbf{v}H - \kappa\nabla H$ express internal redistribution of variance. Here $\mathscr{E} = e_{ij} \equiv \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the deformation tensor. The last terms in (2.7) are positive definite; they evidently represent dissipation of variance.

The connexion between variance and energy merits some comment. After multiplication by $+\frac{1}{2}$, equation (2.7*a*) becomes a balance equation for kinetic energy $\overline{\overline{\frac{1}{2}\mathbf{v}^2}}$. Similarly, after multiplication by $-\frac{1}{2}$, equation (2.7*b*) becomes a balance equation for $\overline{\overline{-\frac{1}{2}R^{-1}T^2}}$, the 'thermobaric' energy (Eckart 1960) or 'available potential' energy (Lorenz 1955). In the energy equations the leading terms on the right are, respectively, $+wT$ and $-wT$; hence, in these equations, wT expresses the rate of *conversion* from thermobaric to kinetic energy, so that in the absence of viscosity, heat conduction and boundary fluxes, the kinetic plus thermal energy $\overline{\overline{\frac{1}{2}(\mathbf{v}^2 - R^{-1}T^2)}}$ is conserved. However, under gravitational *in*stability, the latter expression is not positive definite and therefore cannot be the norm of an orthogonal basis. This explains the relevance of kinetic plus thermal variance $\overline{\overline{\mathbf{v}^2 + R^{-1}T^2}}$ for spectral analysis.

In association with certain boundary conditions the operator $\mathscr{L}$ in (2.4) is self adjoint; that is

$$\{\mathbf{S}, \mathscr{L}\mathbf{S}'\} - \{\mathscr{L}\mathbf{S}, \mathbf{S}'\} = 0. \tag{2.8}$$

To show this, we start from the definition of inner product (2.5), and find that the left side of (2.8) can be written

$$\overline{\overline{\operatorname{div}[\mathbf{B}(\mathbf{S},\mathbf{S}') - \mathbf{B}(\mathbf{S}',\mathbf{S})]}},$$

$$\mathbf{B}(\mathbf{S},\mathbf{S}') \equiv 2\nu\mathbf{v}\,.\,\mathscr{E}' + R^{-1}\kappa T\nabla T'.$$

Therefore, $\mathscr{L}$ is self-adjoint if boundary conditions cause the normal components of $\mathbf{B}(\mathbf{S},\mathbf{S}')$ and $\mathbf{B}(\mathbf{S}',\mathbf{S})$ to vanish on all boundaries. From the definition of $\mathbf{B}$ it is clear that this will be true if the viscous flux $2\nu\mathbf{v}\,.\,\mathscr{E}'$ and the conductive flux $R^{-1}\kappa T\nabla T'$ have zero normal components on all boundaries. These conditions are satisfied in all cases of practical interest; details are given elsewhere (Platzman 1964, pp. 20–4).†

3. The normal modes

If the non-linear terms $\mathbf{v}\,.\,\nabla\mathbf{v}$ and $\mathbf{v}\,.\,\nabla T$ in (2.1) are omitted, there are solutions of the form $(\mathbf{v}_\alpha, T_\alpha, \varpi_\alpha)\exp\sigma_\alpha t$, where $\mathbf{v}_\alpha$, T_α, ϖ_α are continuous space-dependent functions, σ_α is an amplification factor, and α is a wave-number vector. These are the normal modes associated with (2.1) and (2.2). They must satisfy the normal-mode equations

$$\sigma_\alpha \mathbf{v}_\alpha = \nu\nabla^2\mathbf{v}_\alpha + \mathbf{k}T_\alpha - \nabla\varpi_\alpha, \tag{3.1a}$$

$$\sigma_\alpha T_\alpha = \kappa\nabla^2 T_\alpha + Rw_\alpha, \tag{3.1b}$$

$$\nabla\,.\,\mathbf{v}_\alpha = 0, \tag{3.2}$$

and appropriate boundary conditions. In terms of the modal state vector $\mathbf{S}_\alpha \equiv (\mathbf{v}_\alpha, T_\alpha)$ and the modal pressure-gradient vector $\mathbf{G}_\alpha \equiv (\nabla\varpi_\alpha, 0)$, equations (3.1) are

$$\mathscr{L}\mathbf{S}_\alpha = \sigma_\alpha \mathbf{S}_\alpha + \mathbf{G}_\alpha, \tag{3.3}$$

in accordance with the definition of $\mathscr{L}$ in (2.4).

The normal modes $\mathbf{S}_\alpha$ form an orthogonal set, and the normal roots σ_α are real. This depends in the usual way upon the fact that $\mathscr{L}$ is self-adjoint, and also upon the orthogonality of any $\mathbf{S}$ to any $\mathbf{G}$

$$\{\mathbf{S}, \mathbf{G}\} = \overline{\overline{\mathbf{v}\,.\,\nabla\varpi}} = \overline{\overline{\operatorname{div}\mathbf{v}\varpi}} = 0. \tag{3.4}$$

Here the first equality comes from the definition of inner product, the second from the solenoidal property of $\mathbf{v}$, the third from the vanishing of the normal component of $\mathbf{v}$ on all boundaries. Now consider two modal states $\mathbf{S} = \mathbf{S}^*_\alpha$ (complex conjugate of $\mathbf{S}_\alpha$) and $\mathbf{S}' = \mathbf{S}_\beta$. From (2.8) with the aid of (3.3) and (3.4), we find

$$(\sigma_\beta - \sigma^*_\alpha)\{\mathbf{S}^*_\alpha, \mathbf{S}_\beta\} = 0. \tag{3.5}$$

When $R > 0$ the variance $\{\mathbf{S}^*_\alpha, \mathbf{S}_\alpha\}$ is positive definite; therefore $\sigma^*_\alpha = \sigma_\alpha$, so all modes are non-oscillatory under gravitational instability. This proves the well-

† It has been possible and advisable to shorten the present paper by occasional reference to an earlier, more complete version (Platzman 1964). The latter was prepared as a technical report for limited distribution before the present paper was submitted for publication.

known principle of 'exchange of stabilities', whereby free convection appears first as a steady motion, rather than as an oscillation.

Since σ_α and therefore $\mathbf{S}_\alpha$ are real, the asterisks in (3.5) are superfluous. Hence any two modes with distinct roots $\sigma_\alpha \neq \sigma_\beta$ are orthogonal: $\{\mathbf{S}_\alpha, \mathbf{S}_\beta\} = 0$. If $\sigma_\alpha = \sigma_\beta$ but $\mathbf{S}_\alpha \neq \mathbf{S}_\beta$, we can orthogonalize, if necessary, in the usual way. These results are contained in the orthonormality statement

$$\{\mathbf{S}_\alpha, \mathbf{S}_\beta\} = \begin{pmatrix} 0 & \text{if} & \mathbf{S}_\alpha \neq \mathbf{S}_\beta \\ 1 & \text{if} & \mathbf{S}_\alpha = \mathbf{S}_\beta \end{pmatrix}, \tag{3.6}$$

in which the normalization adopted is such that each mode has unit variance.

More specific information about S_α and σ_α is contingent, of course, upon solution of the characteristic-value problem (3.1) and (3.2) with appropriate boundary conditions. We consider now the general features of this problem that are important from the standpoint of spectral analysis.† For this purpose it is necessary to distinguish between three fundamentally different types of solutions of (3.1) and (3.2):

thermal modes	$\mathbf{v} = 0,$	$T \neq 0,$
convective modes	$\mathbf{v} \neq 0,$	$T \neq 0,$
kinetic modes	$\mathbf{v} \neq 0,$	$T = 0.$

(The subscript α is tacit throughout the discussion that follows.)

(*a*) *Thermal modes*

We use the symbol τ in place of T for the temperature field in a thermal mode. Since $\mathbf{v} = 0$ in a thermal mode, (3.2) is satisfied trivially and (3.1*a*) reduces to $\nabla\varpi = \mathbf{k}\tau$, which means that neither ϖ nor τ can be functions of x or y. Therefore, (3.1*b*) reduces to $\kappa D^2\tau = \sigma\tau$, where $D \equiv \partial/\partial z$. If the origin for z is taken midway between the upper and lower boundaries, the solution that meets $\tau = 0$ at $z = \pm\frac{1}{2}$ is

$$\left.\begin{aligned} \tau &= (2R)^{\frac{1}{2}} \begin{pmatrix} \cos n\pi z \ (n \text{ odd}) \\ \sin n\pi z \ (n \text{ even}) \end{pmatrix}, \\ \sigma &= -n^2\pi^2\kappa, \end{aligned}\right\} \tag{3.7}$$

where $n-1 = 0, 1, 2, \ldots$ is the number of horizontal nodal planes, excluding boundaries. The cosine alternative in (3.7) gives symmetric modes (τ symmetric with respect to $z = 0$); the sine gives antisymmetric modes. The normalization factor $(2R)^{\frac{1}{2}}$ is a consequence of the general normalization specified in (3.6). The solution of (3.1) can be completed by solving for ϖ from (3.1*a*) which is now $D\varpi = \tau$, with τ as in (3.7). Thus, the thermal modes can be determined easily and completely; all are damped ($\sigma < 0$).

(*b*) *Convective modes*

A solenoidal vector field $\mathbf{v}$ can be partitioned into a 'poloidal' part $\mathbf{v}_\phi$ and a 'toroidal' part $\mathbf{v}_\psi$ (for example, see Chandrasekhar 1961, p. 622). For slab geometry

$$\mathbf{v}_\phi = -\operatorname{curl}^2 \mathbf{k}\phi = -D\nabla\phi + \mathbf{k}\nabla^2\phi, \tag{3.8a}$$

$$\mathbf{v}_\psi = -\operatorname{curl} \mathbf{k}\psi = \mathbf{k} \times \nabla\psi, \tag{3.8b}$$

† Construction of solutions (analytic and numerical) is summarized elsewhere (Platzman 1964, pp. 82–94).

where $D \equiv \partial/\partial z$. It is evident that $\mathbf{v}_\psi$ and $\mathbf{v}_\phi$ are individually solenoidal; moreover, curl $\mathbf{v}_\phi$ is toroidal and curl $\mathbf{v}_\psi$ is poloidal

$$\operatorname{curl}\mathbf{v}_\phi = \operatorname{curl}(\mathbf{k}\nabla^2\phi) = -\mathbf{k}\times\nabla(\nabla^2\phi),$$
$$\operatorname{curl}\mathbf{v}_\psi = -\operatorname{curl}^2(\mathbf{k}\psi) = -D\nabla\psi + \mathbf{k}\nabla^2\psi.$$

Thus, a poloidal velocity field has horizontal vortex lines but generally three-dimensional streamlines, whereas a toroidal velocity field has horizontal streamlines but generally three-dimensional vortex lines.

We use the symbol θ in place of T for the temperature field in a convective mode. According to (3.8a), $w = \hat{\nabla}^2\phi$ (where $\hat{\nabla}$ is the horizontal gradient operator), so the poloidal velocity can be regarded as determined by w. To determine θ and w, apply $-D\hat{\nabla}$ to the horizontal part of (3.1a), $\hat{\nabla}^2$ to the vertical part, and add the results. This eliminates ϖ; it also eliminates the horizontal velocity vector $\hat{\mathbf{v}}$ because $\hat{\nabla}.\hat{\mathbf{v}} = -Dw$ by (3.2). If (3.1b) is paired with the equation thus obtained, we have

$$\left.\begin{aligned}(\sigma-\nu\nabla^2)\nabla^2 w &= \hat{\nabla}^2\theta,\\ (\sigma-\kappa\nabla^2)\theta &= Rw.\end{aligned}\right\} \tag{3.9}$$

We assume no slip and uniform, constant temperature at horizontal boundaries ($z = \pm\frac{1}{2}$), and conditions of symmetry at vertical boundaries†

$$\left.\begin{aligned}D^2w = w = \theta = 0 &\quad \text{on horizontal boundaries,}\\ \partial w/\partial n = \partial\theta/\partial n = 0 &\quad \text{on vertical boundaries.}\end{aligned}\right\} \tag{3.10}$$

The solution of (3.9) subject to (3.10) has the form (Pellew & Southwell 1940)

$$\left.\begin{aligned}w(x,y,z) &= f(x,y)\,W(z),\\ \theta(x,y,z) &= f(x,y)\,\Theta(z),\end{aligned}\right\} \tag{3.11}$$

where $f(x,y)$ is a characteristic function of the problem

$$-\hat{\nabla}^2 f = a^2 f \tag{3.12}$$

($\partial f/\partial n = 0$ on the vertical boundary), and $W(z)$, $\Theta(z)$ are characteristic functions of the problem

$$\left.\begin{aligned}(D^2-a^2)[\sigma-\nu(D^2-a^2)]\,W &= -a^2\Theta,\\ [\sigma-\kappa(D^2-a^2)]\,\Theta &= RW\end{aligned}\right\} \tag{3.13}$$

($D^2W = W = \Theta = 0$ on horizontal boundaries).

For particular values of parameters a, R, P the characteristic-value problem (3.13) generates a discrete spectrum of admissible values of σ. These values consist of all roots of the *diagnostic equation*

$$\Delta(\sigma;\, a, R, P) = 0. \tag{3.14}$$

This aspect of the σ-spectrum obviously corresponds to the vertical structure of convective modes. In accordance with (3.12) the parameter a itself belongs to a discrete spectrum: this aspect of the σ-spectrum evidently reflects the horizontal

† A full discussion of boundary conditions is given elsewhere (Platzman 1964, pp. 20–24); see also Stuart (1964).

structure of convective modes. Closer analysis of the diagnostic equation reveals that the σ-spectrum consists of two simply denumerable subspectra. One of these subspectra corresponds to convective modes in which average convective heat flux is from hot to cold boundary. We shall call these *free* convective modes, and denote their σ-values by σ_n^+ $(n = 1, 2, 3, \ldots)$. The other subspectrum corresponds to convective modes in which average convective heat flux is from cold to hot boundary. We shall call these *forced* convective modes, and denote their σ-values by σ_n^- $(n = 1, 2, 3, \ldots)$. A forced mode is necessarily damped $(\sigma_n^- < 0)$; a free mode may be amplified, neutral or damped $(\sigma_n^+ > 0, = 0 \text{ or } < 0)$.

According to (3.8*a*), the poloidal velocity has a horizontal part $\hat{\mathbf{v}}_\phi = -\hat{\nabla} D\phi$ and a vertical component $w = \hat{\nabla}^2\phi$. With w as in (3.11), the function ϕ that meets $\hat{\nabla}^2\phi = w$, and the corresponding $\hat{\mathbf{v}}_\phi$, are

$$\phi = -a^{-2} f(x, y) W(z), \tag{3.15}$$

$$\hat{\mathbf{v}}_\phi = -a^{-2}\hat{\nabla} f D W. \tag{3.16}$$

This completes the formal determination of the poloidal part of the convective modes with lateral conditions of symmetry.†

In accordance with (3.8*b*), the toroidal part of the velocity in a convective mode is represented by a stream function ψ. The equation to be satisfied by ψ is obtained by taking the horizontal curl $(\hat{\nabla}\times)$ of (3.1*a*):

$$(\nu\nabla^2 - \sigma)\,\hat{\nabla}^2\psi = 0, \tag{3.17}$$

the equation for vertical component of vorticity. Equation (3.17) obviously contains no reference to the poloidal velocity; and conversely, the differential equation (3.9) (which determines the poloidal velocity) contains no reference to the toroidal velocity. In other words, in a convective mode there is no coupling between poloidal and toroidal components through the differential equations—such as would arise, for example, in a rotating system. Moreover, σ in (3.17) is not a free parameter because its spectrum is determined by the characteristic-value problem associated with the poloidal component.

It follows that a convective mode can have a toroidal component only if the two components are coupled through boundary conditions, in which case (3.17) represents a boundary-value problem. It also can be shown (Platzman 1964, pp. 37–8) that there is no coupling between poloidal and toroidal velocities at horizontal boundaries (rigid or 'free'), and that there is no coupling at vertical polygonal boundaries with conditions of symmetry. Therefore, in convection which tessellates an infinite plane, the velocity field in a convective mode is purely poloidal.‡

In this investigation, detailed numerical results have been obtained only for lateral conditions of symmetry. Since the horizontal velocity of a convective

† A no-slip condition at vertical boundaries does not permit a solution in which the z-co-ordinate can be separated from x and y. Details of this case are given elsewhere (Platzman 1964, pp. 82–3).

‡ If no-slip conditions are imposed on vertical boundaries, the velocity field of a convective mode must have a toroidal component which is coupled to the poloidal component through the lateral boundary conditions—in other words, through the viscous boundary layer on the vertical boundary.

mode is purely poloidal, we have $\hat{\mathbf{v}} = \hat{\mathbf{v}}_\phi$ with $\hat{\mathbf{v}}_\phi$ as in (3.16). If this expression for $\hat{\mathbf{v}}$ is used in conjunction with (3.11) for w and θ, the requirement (3.6) that each mode have unit variance can be stated

$$\langle f^2 \rangle [a^{-2}\overline{(DW)^2} + \overline{W^2} + R^{-1}\overline{\Theta^2}] = 1,$$

where $\langle (\ ) \rangle \equiv \iint (\ )\, dx\, dy$ and $\overline{(\ )} \equiv \int (\ )\, dz$ signify horizontal and vertical integration between the boundaries. (To obtain this result, it is necessary to use the identity $\langle (\hat{\nabla} f)^2 \rangle = a^2 \langle f^2 \rangle$ which comes directly from (3.12).) The preceding equation will be satisfied by choosing

$$\langle f^2 \rangle = 1, \tag{3.18a}$$

$$a^{-2}\overline{(DW)^2} + \overline{W^2} + R^{-1}\overline{\Theta^2} = 1. \tag{3.18b}$$

These are the partial normalizations used subsequently.

(c) *Kinetic modes*

From (3.1 b) it is evident that $w = 0$ when $T = 0$, so in a kinetic mode the velocity field is purely horizontal. Moreover, (3.2) reduces to $\hat{\nabla} . \mathbf{v} = 0$, so $\mathbf{v}$ can be represented by a stream function: $\mathbf{v} = \mathbf{k} \times \nabla \psi$. In other words, the velocity field of a kinetic mode is purely toroidal. The analysis of this field is similar to that for the toroidal part of a convective mode.† In particular, the differential equation for ψ is (3.17). All kinetic modes are damped.

4. The spectral-dynamic equations

Consider an arbitrary state $\mathbf{S}$ of finite-amplitude convection, which satisfies the governing equation (2.4) for specified R and P, and appropriate boundary conditions. Let $\mathbf{S}_\alpha$ be the normal modes associated with the same values of R and P and the same boundary conditions. Subject to quadratic integrability of $\mathbf{S}$ and completeness of the $\mathbf{S}_\alpha$, we have the expansion

$$\mathbf{S}(x,y,z,t) = \sum_\alpha A_\alpha(t)\, \mathbf{S}_\alpha(x,y,z), \quad A_\alpha \equiv \{\mathbf{S}_\alpha, \mathbf{S}\}, \tag{4.1}$$

in view of the orthonormality (3.6). Moreover, the associated Parseval relation

$$\{\mathbf{S}, \mathbf{S}\} = \sum_\alpha A_\alpha^2 \tag{4.2}$$

shows that A_α^2 constitutes the variance spectrum of $\mathbf{S}$. If the spectrum is not completely discrete, the sums over α should be interpreted as including integration over continuous portions of the spectrum. However, we need be concerned here only with states of $\mathbf{S}$ which can be represented by a completely discrete spectrum, because the normal-mode equations (3.3) are free of singularities, and because boundary conditions are applied over a finite region.

The governing equation (2.4) can be transformed to the domain of the expansion coefficients A_α as follows. If $A_\alpha \equiv \{\mathbf{S}_\alpha, \mathbf{S}\}$ is differentiated with respect to t, and $\partial \mathbf{S}/\partial t$ then replaced by (2.4), we have

$$dA_\alpha/dt = \{\mathbf{S}_\alpha, \mathscr{L}\mathbf{S}\} - \{\mathbf{S}_\alpha, \mathbf{v} . \nabla \mathbf{S}\},$$

† Details are given elsewhere (Platzman 1964, pp. 40–3).

since $\{\mathbf{S}_\alpha, \mathbf{G}\} = 0$. The first term can be rewritten with the help of (2.8) and (3.3), and the fact that $\{\mathbf{G}_\alpha, \mathbf{S}\} = 0$. The preceding result is then

$$dA_\alpha/dt = \sigma_\alpha A_\alpha - \{\mathbf{S}_\alpha, \mathbf{v}\,.\,\nabla\mathbf{S}\}. \tag{4.3}$$

Thus far, the transformation depends upon existence of the A_α, but not upon validity of the expansion (4.1). Assume now that this expansion is valid, and substitute expansions for $\mathbf{v}$ and $\mathbf{S}$ in the convection term $\mathbf{v}\,.\,\nabla\mathbf{S}$; then

$$dA_\alpha/dt = \sigma_\alpha A_\alpha - \sum_\beta \sum_\gamma A_\beta A_\gamma \{\mathbf{S}_\alpha, \mathbf{v}_\beta\,.\,\nabla\mathbf{S}_\gamma\}. \tag{4.4}$$

The *spectral-dynamic equation* (4.4) poses an initial-value problem for the expansion coefficients, in terms of an infinite system of quasi-linear, ordinary time-differential equations of first-order with constant coefficients. It expresses the 'speed' of each expansion coefficient as the sum of a linear, modal term $\sigma_\alpha A_\alpha$ (exponential effect), and a non-linear, convection term which is a quadratic form over all coefficients. The 'coupling integrals' $\{\mathbf{S}_\alpha, \mathbf{v}_\beta.\nabla\mathbf{S}_\gamma\}$ of this quadratic form are completely determined by the structure of the normal modes, and therefore are known in principle. In a steady state of convection, the spectral-dynamic equation reduces to

$$\sigma_\alpha A_\alpha = \sum_\beta \sum_\gamma A_\beta A_\gamma \{\mathbf{S}_\alpha, \mathbf{v}_\beta\,.\,\nabla\mathbf{S}_\gamma\}, \tag{4.5}$$

an infinite set of quadratic, algebraic equations for the A_α.

It is illuminating to examine the spectral form of the balance equation for variance $\{\mathbf{S}, \mathbf{S}\}$. Thus, if (4.3) is multiplied by A_α and summed over α, the convection term becomes $\{\mathbf{S}, \mathbf{v}\,.\,\nabla\mathbf{S}\}$, which is zero because of the solenoidal character of $\mathbf{v}$; hence

$$\frac{d}{dt}\tfrac{1}{2}\sum_\alpha A_\alpha^2 = \sum_\alpha \sigma_\alpha A_\alpha^2. \tag{4.6}$$

It is clear now that in the spectral-dynamic equation (4.3) or (4.4) the linear, modal term $\sigma_\alpha A_\alpha$ represents *generation* of variance, whereas the non-linear, convection term represents *exchange* of variance across the spectrum. An amplified mode contributes positive generation of variance, a damped mode negative generation. It sometimes has been suggested that a steady state of convection can exist for subcritical values of the Rayleigh number provided the amplitude of convection is sufficiently large. Equation (4.6) shows that this is not possible, for in a subcritical state all modes are damped, so the right side of (4.6) is negative definite. The variance in any subcritical state therefore is a monotone decreasing function of time.†

The complete spectral-dynamic equations cannot be solved in closed form. However, any state of finite-amplitude convection can be approximated by truncating the spectrum of this state; that is, by assigning zero values to all expansion coefficients except those belonging to a selected group of dominant modes. Let δ signify the wave-number vectors of modes in the truncated spectrum, and let $\dot{\mathbf{S}}$ be an approximation to $\mathbf{S}$ that spans δ

$$\dot{\mathbf{S}}\,(x,y,z,t) = \sum_\delta \dot{A}_\delta(t)\,\mathbf{S}_\delta(x,y,z), \quad \dot{A}_\delta \equiv \{\mathbf{S}_\delta, \dot{\mathbf{S}}\}. \tag{4.7}$$

† An equivalent proof has been given by Howard (1963).

The approximation $\dot{\mathbf{S}}$ will be specified now by asserting that the $\dot{A}_\delta$ satisfy the *truncated* spectral-dynamic equation

$$d\dot{A}_\delta/dt = \sigma_\delta \dot{A}_\delta - \{\mathbf{S}_\delta, \dot{\mathbf{v}}.\nabla\dot{\mathbf{S}}\}, \tag{4.8}$$

in place of (4.3). Equation (4.8) is formally identical to (4.3), but in fact differs from (4.3) significantly because in the convection term of (4.8) each of the factors $\dot{\mathbf{v}}$ and $\dot{\mathbf{S}}$ spans only the *truncated* spectrum. If the truncation is severe, it is possible that (4.8) can be solved for the $\dot{A}_\delta$.

The nature of the approximation $\dot{\mathbf{S}}$ determined by (4.8) can be understood from the result of multiplying (4.8) by $\mathbf{S}_\delta$ and summing over δ; since $\sigma_\delta \mathbf{S}_\delta = \mathscr{L}\mathbf{S}_\delta - \mathbf{G}_\delta$, we get

$$\partial\dot{\mathbf{S}}/\partial t = \mathscr{L}\dot{\mathbf{S}} - (\dot{\mathbf{v}}.\nabla\dot{\mathbf{S}})_m - \dot{\mathbf{G}},$$
$$(\dot{\mathbf{v}}.\nabla\dot{\mathbf{S}})_m \equiv \sum_\delta \{\mathbf{S}_\delta, \dot{\mathbf{v}}.\nabla\dot{\mathbf{S}}\}\,\mathbf{S}_\delta,$$

where $\dot{\mathbf{G}} \equiv \sum_\delta \dot{A}_\delta G_\delta$ and $(\dot{\mathbf{v}}.\nabla\dot{\mathbf{S}})_m$ denotes an approximation-in-the-mean to $\dot{\mathbf{v}}.\nabla\dot{\mathbf{S}}$, namely, the approximation that spans the set δ. Thus, although the spectrum of $\dot{\mathbf{S}}$ is limited to δ, that of the quadratic form $\dot{\mathbf{v}}.\nabla\dot{\mathbf{S}}$ is not. To this extent the approximation $\dot{\mathbf{S}}$ fails to satisfy the governing equation (2.4). It should be noted that $\dot{\mathbf{S}}$ is *not* an approximation-in-the-mean to $\mathbf{S}$, in the sense that for the specified truncation the residual variance $\{\mathbf{S}-\dot{\mathbf{S}}, \mathbf{S}-\dot{\mathbf{S}}\}$ is a minimum with respect to $\dot{A}_\delta$. In fact, the $\dot{A}_\delta$ determined by (4.8) are such that they are altered, in general, by altering the set δ, so they are not the true expansion coefficients of $\mathbf{S}$. Nevertheless, we shall adopt the working assumption that close approximations to the true expansion coefficients can be obtained from truncated spectral-dynamic equations if the truncation is made in a reasonable way. We turn next to some considerations that form the basis for the truncation procedure adopted in this investigation.

The distinction has been made previously between thermal modes ($\mathbf{v}_\alpha = 0$, $T_\alpha = \tau_\alpha \neq 0$), convective modes ($\mathbf{v}_\alpha \neq 0$, $T_\alpha = \theta_\alpha \neq 0$), and kinetic modes ($\mathbf{v}_\alpha \neq 0$, $T_\alpha = 0$). For completeness the expansion (4.1) must span all three modal species; thus, in general $\mathbf{S} = \mathbf{S}' + \mathbf{S}'' + \mathbf{S}'''$, where

$$\mathbf{S}' \equiv \begin{pmatrix} 0 \\ \tau \end{pmatrix}, \quad \mathbf{S}'' \equiv \begin{pmatrix} \mathbf{v}'' \\ \theta \end{pmatrix}, \quad \mathbf{S}''' \equiv \begin{pmatrix} \mathbf{v}''' \\ 0 \end{pmatrix} \tag{4.9}$$

are, respectively, the parts of $\mathbf{S}$ that span thermal, convective and kinetic modes. The structure of thermal and convective modes is such that $\tau = \tau(z,t)$, whereas $\langle\theta\rangle = 0$ in view of (3.12). Thus the partition $T = \tau + \theta$ is the same as partition of T into a horizontal average $\tau = \langle T\rangle$ and deviation $\theta = T - \langle T\rangle$.

Any realistic approach to truncation of the spectral-dynamic equation must begin with partition of this equation into the three equations that govern the spectral dynamics of each of the three modal species. Using primes to identify these species, in the sense indicated above, and starting from (4.3), we have

$$dA'_\alpha/dt = \sigma'_\alpha A'_\alpha - \{\mathbf{S}'_\alpha, \mathbf{v}.\nabla\mathbf{S}\},$$
$$dA''_\alpha/dt = \sigma''_\alpha A''_\alpha - \{\mathbf{S}''_\alpha, \mathbf{v}.\nabla\mathbf{S}\},$$
$$dA'''_\alpha/dt = \sigma'''_\alpha A'''_\alpha - \{\mathbf{S}'''_\alpha, \mathbf{v}.\nabla\mathbf{S}\}.$$

Consider the convection term in the first of these equations

$$\{\mathbf{S}'_\alpha, \mathbf{v}\,.\,\nabla\mathbf{S}\} = R^{-1}\overline{\overline{\tau_\alpha \mathbf{v}\,.\,\nabla T}} = R^{-1}\overline{\tau_\alpha\langle\nabla\,.\,\mathbf{v}T\rangle}$$
$$= R^{-1}\overline{\tau_\alpha D\langle wT\rangle} = R^{-1}\overline{\tau_\alpha D\langle w\theta\rangle}.$$

The first equality involves the definition of the inner product and the fact that $\mathbf{v}'_\alpha = 0$; the second comes from the facts that τ_α is a function of z only and that $\mathbf{v}$ is solenoidal; the third is a result of $\mathbf{v}$ having zero normal component on lateral boundaries; and the fourth makes use of the fact that τ is a function of z only, whereas $\langle w\rangle = 0$ (in view of (3.12)). (It should be noted that $w = w''$, since $w''' = 0$ owing to the toroidal character of $\mathbf{v}'''$.)

After similar transformations are made in the other convection terms, the partitioned spectral-dynamic equations can be written

$$dA'_\alpha/dt = \sigma'_\alpha A'_\alpha - R^{-1}\overline{\tau_\alpha D\langle w\theta\rangle}, \tag{4.10a}$$

$$dA''_\alpha/dt = \sigma''_\alpha A''_\alpha - R^{-1}\overline{\langle w\theta_\alpha\rangle D\tau} + R^{-1}\overline{\overline{\theta_\alpha(\mathbf{v}\,.\,\nabla)\,\theta}}$$
$$- \overline{\overline{\mathbf{v}''_\alpha\,.\,(\mathbf{v}\,.\,\nabla)\,\mathbf{v}''}} - \overline{\overline{\mathbf{v}''_\alpha\,.\,(\mathbf{v}\,.\,\nabla)\,\mathbf{v}'''}}, \tag{4.10b}$$

$$dA'''_\alpha/dt = \sigma'''_\alpha A'''_\alpha - \overline{\overline{\mathbf{v}'''_\alpha\,.\,(\mathbf{v}\,.\,\nabla)\,\mathbf{v}'''}} - \overline{\overline{\mathbf{v}'''_\alpha\,.\,(\mathbf{v}\,.\,\nabla)\,\mathbf{v}''}}. \tag{4.10c}$$

The seven convection terms in (4.10) can be classified as corresponding to *internal exchange* or *external exchange* of variance. Thus, if the three equations in (4.10) are multiplied, respectively, by A'_α, A''_α, A'''_α and then summed, respectively, over the portion of the spectrum spanned by each of the three modal species, we obtain balance equations for the corresponding parts of the variance spectrum. These equations need not be written explicitly, but it is clear that as a result of the operations just described, the second and third convection terms in (4.10*b*) are nugatory, owing to the solenoidal character of $\mathbf{v}$. These terms therefore express internal exchange of variance within $\mathbf{S}''$. Similarly, the first convection term in (4.10*c*) expresses internal exchange within $\mathbf{S}'''$. The remaining four convection terms evidently represent external exchange of variance, that is, exchange between two different parts of the spectrum. Thus, the convection term in (4.10*a*) and the first in (4.10*b*) express exchange between $\mathbf{S}'$ and $\mathbf{S}''$, and the last terms in (4.10*b*) and (4.10*c*) express exchange between $\mathbf{S}''$ and $\mathbf{S}'''$.

The leading terms on the right in (4.10) represent generation of variance within each of the three parts of the spectrum. It is evident that variance generation by thermal and kinetic modes is negative definite, because all of these modes are damped. Hence, a steady state can exist only if variance is supplied by external exchange from the convective part of the spectrum to the thermal and kinetic parts. (Note that there is no external exchange between kinetic and thermal modes.) These circumstances are portrayed schematically in figure 1.

Inspection of (4.10*a*) confirms that the thermal part of the variance spectrum is not affected by internal exchange. (This results from the fact that the governing equations are bilinear, rather than quadratic, in temperature.) In other words, there is no *explicit* non-linearity in $(4.10a)$ with respect to expansion coefficients A'_α. Consequently, this equation can be 'solved' as follows:

$$A'_\alpha(t) = A'_\alpha(0)\exp\sigma'_\alpha t - R^{-1}\int_0^t \overline{\tau_\alpha D\langle w\theta\rangle}\exp\sigma'_\alpha(t-T)\,dT. \tag{4.11}$$

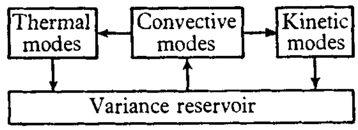

FIGURE 1. Schematic representation of variance balance between thermal, convective and kinetic modes. Horizontal arrows represent exchange of variance, vertical arrows generation of variance. Directions are those appropriate for a steady state.

If this expression is multiplied by τ_α and then summed over the spectrum of thermal modes, the result is an explicit formula for $\tau(z,t)$ in terms of the initial spectrum $A'_\alpha(0)$ of thermal modes and the function $\langle w\theta\rangle$, which depends only upon the convective part of the spectrum. Another way to arrive at the same result is to note that $(4.10a)$ is the spectral form of

$$\partial\tau/\partial t = \kappa D^2\tau - D\langle w\theta\rangle, \tag{4.12}$$

the horizontal average of the governing equation $(2.1b)$,with $\tau = \langle T\rangle$. If (4.12) is solved for $\tau(z,t)$ by means of an appropriate Green's function, the term $D\langle w\theta\rangle$ being regarded as an inhomogeneous part of the equation, the result will be the same as that deduced from (4.11).

If $D\tau$ thus determined is inserted in the first convection term of $(4.10b)$, *the thermal part of the spectrum will be completely eliminated*, and the spectral-dynamic equations will be 'closed' on the convective and kinetic modes. (It should be noted that after elimination of $D\tau$ in this manner, the first convection term is *cubic* in the convective expansion coefficients A''_α.) For unsteady convection this elimination brings $(4.10b)$ into the form of a differential-integral equation, but in a steady state the equations are algebraic and the elimination leads to a simple result. In a steady state (4.12) implies that

$$\kappa D\tau = \langle w\theta\rangle + \text{const.}$$

The constant can be evaluated from the fact that $\overline{D\tau} = 0$ since $\tau = 0$ at horizontal boundaries; therefore

$$\kappa D\tau = \langle w\theta\rangle - \langle\overline{w\theta}\rangle \quad \text{(steady state)} \tag{4.13}$$

a well-known fundamental relation. Equation (4.13) can be integrated in principle, to obtain τ; but in fact for elimination in $(4.10b)$ we need $D\tau$ rather than τ.

The foregoing analysis can be summarized as follows. We have seen that any state of convection can be partitioned into three parts

$$\mathbf{S}'\,(\text{thermal}) + \mathbf{S}''\,(\text{convective}) + \mathbf{S}'''\,(\text{kinetic}),$$

each of which spans one of the three distinct modal species. Variance of $\mathbf{S}'$ and $\mathbf{S}''$ is always dissipated because thermal and kinetic modes are wholly damped. In a steady state, variance balance is maintained through generation within $\mathbf{S}''$ by self-excited convective modes, and transfer from $\mathbf{S}''$ to $\mathbf{S}'$ and $\mathbf{S}'''$ (figure 1). It is apparent that $\mathbf{S}''$ plays a crucial role in this process; and of course it is within $\mathbf{S}''$ that the convective flux of heat is accomplished. Owing to the fact that the spectrum of $\mathbf{S}'$ cannot be altered by internal exchange, the spectral-dynamic equation for $\mathbf{S}'$ is linear and can be solved explicitly. This makes possible the complete elimination of $\mathbf{S}'$, so that the spectral-dynamic equations can be closed on $\mathbf{S}''$ and $\mathbf{S}'''$. Accordingly, the procedure that we shall follow for spectrum truncation is to retain the complete spectrum of $\mathbf{S}'$, while truncating the spectrum of $\mathbf{S}''$ and that of $\mathbf{S}'''$ very severely.

5. The spectrum of plane tessellation

We examine briefly the characteristic-value spectrum of convective modes under tessellation of the infinite plane. This consists of the roots of the diagnostic equation (3.14), in which a, R, P are parameters. It has already been pointed out that the roots of this equation can be divided into two groups, denoted σ_n^+ and σ_n^- $(n = 1, 2, 3, \ldots)$: the former corresponds to free modes, the latter to forced modes (§ 3). We restrict attention now to the free convective modes with maximum σ_n^+ for given a, R, P. Call these the *primary* convective modes. Assuming the index n to be chosen so that $\sigma_1^+ > \sigma_2^+ > \sigma_3^+ > \ldots$, the root that belongs to the modes in question is σ_1^+. Since the horizontal wave-number a itself belongs to a discrete spectrum, namely the characteristic-value spectrum of (3.12), the primary convective modes and the associated values of σ_1^+ form a discrete sub-spectrum of free convective modes (for given R, P).

The diagnostic equation may be portrayed by means of the familiar diagram with axes R, a—in which the roots σ are represented as functions of R and a (for fixed P) by level curves (diagnostics) $\sigma = \text{const}$. An example of such a diagram is shown in figure 2.† In figure 3 the curve marked $\sigma_1^+ = 0$ is (schematically) the diagnostic of marginal stability for primary convective modes. To the left of $\sigma_1^+ = 0$ these modes are damped, to the right they are amplified. Hence for $R < R_c$ (the minimum R for which the primary can be amplified) *all* free modes are damped and a steady state is not possible. If $R > R_c$, a finite-amplitude steady convection is possible, in principle, in which variance balance is maintained through self-excitation of the primary mode.

Consider now the problem of plane tessellation, in which the finite-amplitude convection is steady and is arranged in a periodic cellular pattern over an infinite plane. For the sake of being explicit in the simplest possible terms, the argument will be illustrated by the case of infinite rolls, with lateral cell boundaries formed by vertical parallel planes. If L is the distance between successive planes, the admissible values of a in this case are integral multiples ka_1 $(k = 1, 2, 3, \ldots)$ of a fundamental value $a_1 = \pi/L$ determined solely by the linear dimension L that is characteristic of the tessellation considered. We shall call the primary mode

† Figure 2 is discussed elsewhere (Platzman 1964, pp. 91–4).

associated with a_1 the *fundamental* primary mode. The definition of L is such that the fundamental primary mode must be the mode that becomes self-excited at $R = R_c$, since otherwise L could not be the distance between successive cell walls. This means that at marginal stability $a_1 = a_c$ and $a_k = ka_c$, as pictured in figure 3. For an infinitesimally supercritical value of R, we may postulate a steady convection in which only the fundamental primary mode is self-excited and all other

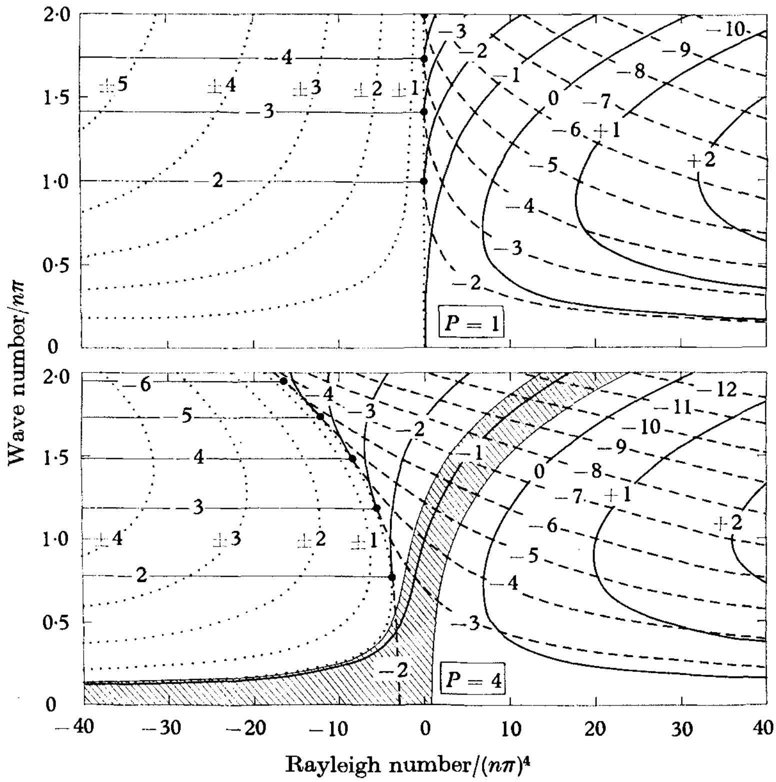

FIGURE 2. Diagnostic diagrams for 'free' boundaries, in terms of scaled parameters $R/n^4\pi^4$, $a/n\pi$ and $\sigma/n^2\pi^2$. Solid curves are diagnostics of free modes (labelled with values of $\sigma^+/n^2\pi^2$); broken curves are diagnostics of forced modes (labelled with values of $\sigma^-/n^2\pi^2$). Dotted curves are contours of Im $\sigma/n^2\pi^2$ in the region of oscillatory modes; contours of Re $\sigma/n^2\pi^2$ in this region are horizontal straight lines. The boundary of the region of oscillatory modes is indicated by heavy dots (the envelope of diagnostics of non-oscillatory modes).

modes are maintained by transfer across the spectrum. Thus, if R is increased quasi-statically above the critical value R_c, the finite-amplitude convection at each R presumably is a pattern of tessellation with a unique scale L that in some manner is determined by the prevailing value of R. On this view, the fundamental scale $a_1 = \pi/L$ is a definite function of R, such as is illustrated schematically by the lower broken curve in figure 3.

The preceding discussion suggests that in a physically realizable process in which R increases quasi-statically through moderately supercritical states, each wave-number a_k in the a-spectrum of primary modes will move along a characteristic curve $ka_1(R)$ determined by that of the fundamental $a_1(R)$. However, as is well known, if steady, finite-amplitude tessellation is assumed at the outset, the curve $a_1(R)$ is indeterminate within the frame-work of the problem—a lack of uniqueness which must be expected when steady solutions are sought in an infinite domain. Uniqueness might be established on a deductive basis by

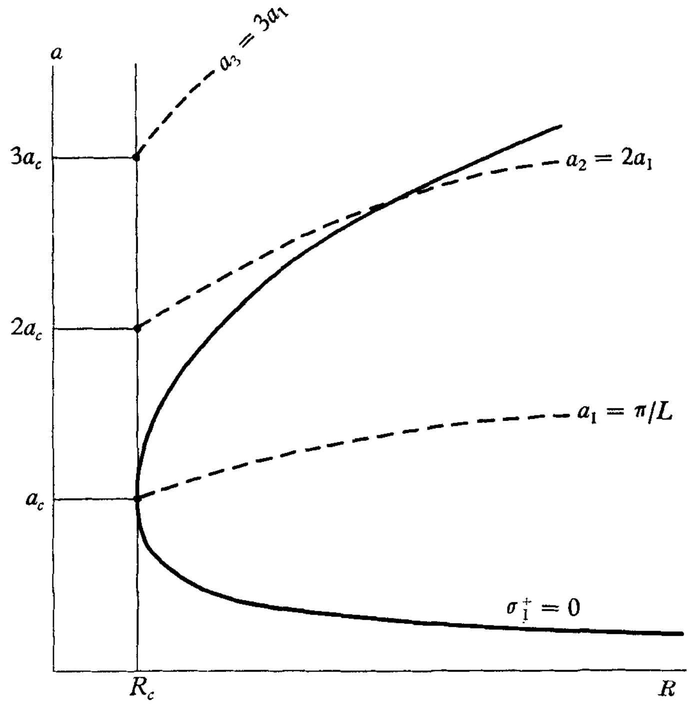

FIGURE 3. Schematic diagnostic diagram. Solid curve is diagnostic of marginal stability for primary convective modes. Broken curves are characteristic curves for horizontal wave-number spectrum in the case of plane tessellation in infinite rolls.

relaxing the requirement that the solution be steady, or that the domain be infinite, or both. However, for numerical computation of heat flux in this investigation (§ 6), uniqueness is established through two alternative *ad hoc* procedures. From a numerical standpoint, the simpler of these is based upon the supplementary condition

$$\sigma_1^+ = \text{maximum with respect to } a, \tag{5.1}$$

which would be valid if the horizontal scale of the fundamental primary mode tends to the value for which this mode is in its linearly most unstable configuration. This procedure involves repeated solution of the diagnostic equation (3.14) for the root σ_1^+, with a sequence of values of a aimed at convergence on the maximum σ_1^+ (for specified R and P). The alternative procedure used is based upon maximization of the heat flux itself. (This would be valid if the horizontal scale tends to the one for which the rate of entropy production is a maximum.) The supplementary condition in this case is

$$N = \text{maximum with respect to } a, \tag{5.2}$$

where N is the Nusselt number. This procedure involves much more computation than (5.1), because it requires not only repeated solution of the diagnostic equation, but also repeated construction of the functions $W(z)$, $\Theta(z)$. Nevertheless, it is entirely feasible by means of a high-speed computer.

Although not needed for the heat-flux computation to be described later, a few remarks will be made here about the spectrum of hexagonal tessellation, because of their relevance to the general problem of spectral analysis by normal modes. Let L be the side length of the equilateral hexagons that form the cell boundary, and place (x, y)-axes so that the origin coincides with the centre of the hexagon whose sides are given by

$$(x/\sqrt{3} \pm \tfrac{1}{2}L)\,(x/\sqrt{3}+y \pm L)\,(x/\sqrt{3}-y \pm L) = 0.$$

Then (3.12) generates the set of orthogonal functions

$$\hat{f}_p(x,y) = \frac{1}{6}\sum_{k=0}^{2}\{\cos \operatorname{Re}[p\, e^{\frac{2}{3}ik\pi}(x+iy)] + \cos \operatorname{Re}[p^*\, e^{\frac{2}{3}ik\pi}(x+iy)]\}, \qquad (5.3)$$

$$a_p = |p| = (3l^2+m^2)^{\frac{1}{2}}\, 4\pi/3L,$$

$$p \equiv (l\sqrt{3}+im)\, 4\pi/3L.$$

Here l and m are any two integers or halves of any two odd integers; in other words, $2l$ and $2m$ are any two integers having the same parity. Note that (5.3) has hexagonal symmetry in the plane of the wave-number vector p, as well as in the (x, y)-plane; i.e. $\hat{f}_p(x, y)$ is not altered by rotation of p into $p\, e^{\frac{1}{3}i\pi}$ or of $x+iy$ into $(x+iy)\, e^{\frac{1}{3}i\pi}$. The vector p is two-dimensional, as required for completeness of the set $\hat{f}_p$. The normalization in (5.3) has been chosen to make $\hat{f} = 1$ for $x+iy = 0$. The function

$$f_p(x,y) \equiv 2\epsilon_p\, 3^{-\frac{1}{4}} L^{-1} \hat{f}_p(x,y)$$

is normalized so that $\langle f^2 \rangle = 1$, in conformity with (3.18a). Here $\epsilon_p = 1$ if $\arg p = 0 \bmod \frac{1}{6}\pi$; otherwise $\epsilon_p = 2^{\frac{1}{2}}$.

From the standpoint of spectral analysis of the variance cascade associated with horizontal structure, an important aspect of the set f_p is the spectrum of the product $f_p f_{p'}$ where p and p' are two generally distinct vectors. By straightforward application of (5.3), it is possible to show that this product spans twelve members of the set. To express this result conveniently, we write $f(p)$ instead of $f_p(x, y)$, on the understanding that x, y are held fixed. The multiplication formula is

$$\hat{f}(p)\hat{f}(p') = \frac{1}{12}\sum_{k=0}^{5}[\hat{f}(p+p'\, e^{\frac{1}{3}ik\pi}) + \hat{f}(p^*+p'\, e^{\frac{1}{3}ik\pi})]. \qquad (5.4)$$

In the light of the fact that $\hat{f}(p)$ is not changed by substitution of p^* or $p\, e^{i\pi/3}$ for p, it can be seen that the right-hand side of (5.4) is symmetric in p and p'.

Christopherson (1940) gave the special solution that corresponds to $l = 0$ in (5.3). However, for spectral analysis of hexagonal tessellation, it is necessary to build a two-dimensional spectrum. For example, consider the fundamental hexagon $(l, m) = (0, 1)$ with wave-number vector $p = i4\pi/3L$. The most important interaction within the horizontal wave-number spectrum is that of $(0, 1)$ with itself. It is easy to see from (5.4) (with $p' = p = i\,4\pi/3L$) that three distinct modes are generated by this interaction, namely $(0, 0)$, $(0, 2)$ and $(\frac{1}{2}, \frac{3}{2})$. The first corresponds to $f = \text{const.}$ and the second to the lowest submultiple of the primary. The latter is a hexagon with vertices at the same azimuths as the fundamental, but linear scale half that of the fundamental. (These two modes are analogous to the types that would be expected in self-interaction of a 'cosine' mode in rectangular tessellation.) The third is not a submultiple of the fundamental in the usual sense—that is, it is not of the form $(0, m)$. It corresponds to the hexagon obtained by rotating the fundamental through an angle of 30°, and reducing its scale by the factor $\sqrt{3}$.

6. The heat flux by primary convective modes

The most rudimentary truncation having physical interest is one in which the convective part of the spectrum is reduced to a single mode, and all kinetic modes are discarded. Let ϵ denote the wave-number vector of the convective mode that is retained. Then $\theta = A_\epsilon \theta_\epsilon$ and $\mathbf{v}'' = A_\epsilon \mathbf{v}_\epsilon$, so in (4.10$b$) (with $\alpha = \epsilon$) the second and third convection terms disappear. The fourth also disappears because $\mathbf{v}''' = 0$ in the truncation being considered. Since $w = A_\epsilon w_\epsilon$, the spectral-dynamic equation (4.10b) for A_ϵ becomes, in the steady state,

$$\overline{\langle w_\epsilon \theta_\epsilon \rangle D\tau} = R\sigma_\epsilon, \tag{6.1}$$

after removal of the factor A_ϵ. We now eliminate $D\tau$ by means of (4.13), which in the case being considered reduces to

$$\kappa D\tau = (\langle w_\epsilon \theta_\epsilon \rangle - \overline{\langle w_\epsilon \theta_\epsilon \rangle}) A_\epsilon^2. \tag{6.2}$$

After elimination of $D\tau$ from (6.1), we may solve for A_ϵ^2

$$\left.\begin{aligned} A_\epsilon^2 &= \kappa R\sigma_\epsilon / \overline{(Q_\epsilon - \overline{Q}_\epsilon)^2}, \\ Q_\epsilon &\equiv \langle w_\epsilon \theta_\epsilon \rangle = W_\epsilon(z)\, \Theta_\epsilon(z). \end{aligned}\right\} \tag{6.3}$$

Here Q_ϵ is a z-dependent function proportional to the upward convective flux of heat by the ϵ-mode. The introduction of W_ϵ and Θ_ϵ is in accord with the normal-mode solution (3.11) and normalization (3.18a). To complete the solution of (6.1) and (6.2), it is necessary to determine the function $\tau(z)$ that spans the thermal modes. By insertion of (6.3) into (6.2); we obtain

$$D\tau = R\sigma_\epsilon (Q_\epsilon - \overline{Q}_\epsilon) / \overline{(Q_\epsilon - \overline{Q}_\epsilon)^2}, \tag{6.4}$$

from which $\tau(z)$ can be obtained by quadrature.

Equations (6.3) and (6.4) give the explicit steady solution of the truncated spectral-dynamic equations in the case of truncation to a single convective mode, in terms of the associated growth rate σ_ϵ and vertical functions W_ϵ, Θ_ϵ. It is clear from (6.3) that this solution is meaningless unless the selected convective mode is in a self-excited state ($\sigma_\epsilon > 0$). Moreover, truncation to a single, self-excited convective mode is realistic only if the mode in question is a *primary* convective mode, that is, a mode having maximum σ_ϵ for given a, R, P. It also is important to stress that σ_ϵ, W_ϵ, Θ_ϵ in solutions (6.3) and (6.4) represent structure of the self-excited mode at the *supercritical* value of the Rayleigh number for which these solutions are to be evaluated.

The result obtained in (6.3) may be used to determine the heat flux associated with this rudimentary truncation. The total upward heat flux across an arbitrary horizontal plane is, in dimensional form,

$$N' \equiv -k'D'\langle\vartheta'\rangle + \rho_0' c' \langle w'\vartheta'\rangle$$

(conduction plus convection), where k' is the thermal conduction coefficient and c' a specific heat. If the dimensions are eliminated through division by

$$k'\Delta\vartheta'/d' = k'R',$$

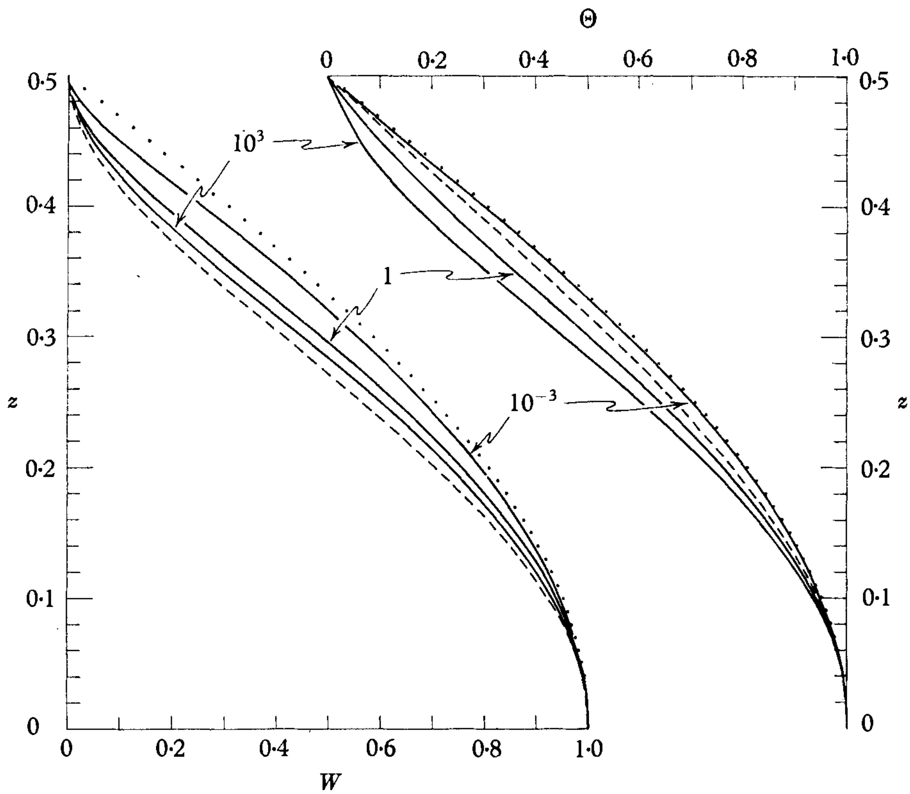

FIGURE 4. Characteristic functions $W(z)$ and $\Theta(z)$ for rigid boundaries, normalized to unity at $z = 0$. Broken curves are for marginal value $R = R_c$ (and are independent of P). Solid curves are for maximum σ at $R = 10^5$, and for three values of P, as indicated. Dotted curves are for 'free' boundaries (and are independent of R and P).

the result is a Nusselt number

$$N = 1 - R^{-1}D\tau + (\kappa R)^{-1}\langle w\theta\rangle, \tag{6.5}$$

where κ is the dimensionless thermal diffusivity $\kappa' = k'/\rho_0' c'$. (To obtain (6.5) the partition $\vartheta = \vartheta_0 - Rz + \tau + \theta$ was introduced, and also the fact that $\langle w\rangle = 0$.) In a steady convection $D\tau$ is given by (4.13); this makes

$$N = 1 + (\kappa R)^{-1}\langle\overline{w\theta}\rangle \quad \text{(steady state)} \tag{6.6}$$

a well-known formula.

Equation (6.6) is an exact expression for heat flux in steady convection. In the case of truncation to a single convective mode, $\langle w\theta\rangle = \langle w_\epsilon \theta_\epsilon\rangle A_\epsilon^2$; hence with A_ϵ^2 as in (6.3)

$$N = 1 + \sigma_\epsilon \overline{Q}_\epsilon / \overline{(Q_\epsilon - \overline{Q}_\epsilon)^2}, \tag{6.7}$$

where $Q_\epsilon \equiv W_\epsilon \Theta_\epsilon$. By direct numerical analysis of (3.13), the functions $W(z)$ and $\Theta(z)$ were computed for a wide range of values of R and P, with wave-number a

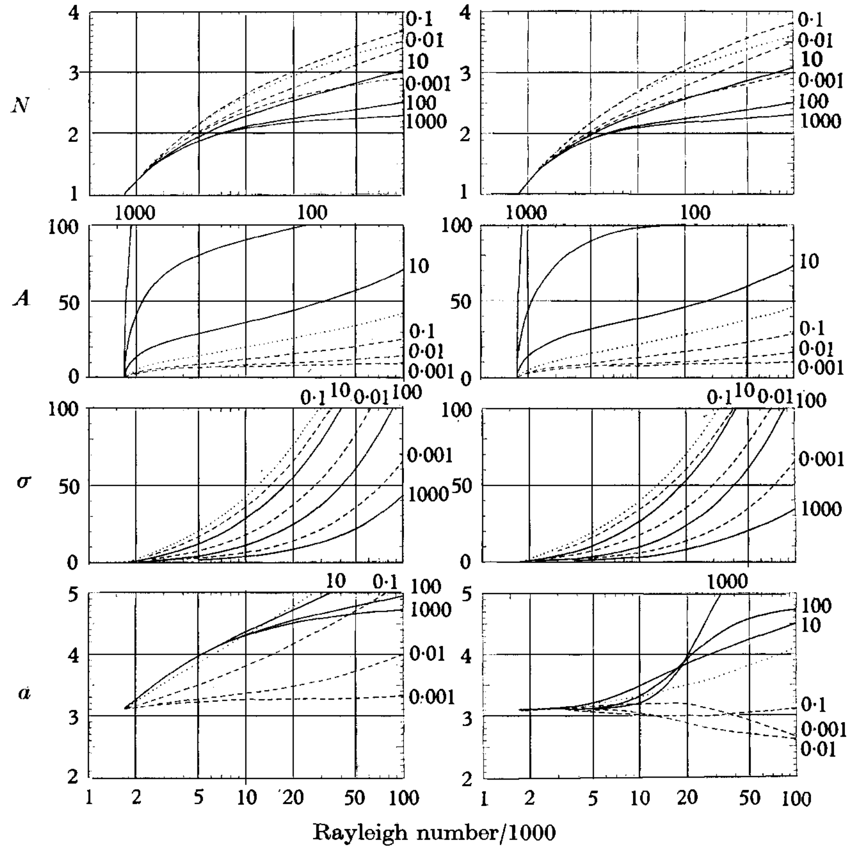

FIGURE 5. Results of numerical computation of heat flux (N), spectral amplitude (A), amplification factor (σ), and horizontal wave number (a). Left panel: maximization of σ; right panel: maximization of N. Prandtl numbers are shown along upper and right borders of each diagram. Solid curves are for $P > 1$, dotted for $P = 1$, broken for $P < 1$. Heat flux is based upon spectral truncation to one convective mode.

selected on the basis of alternative *ad hoc* uniqueness procedures (5.1) and (5.2). Figure 4 gives a representative selection of results. In this figure the functions are normalized to unity at $z = 0$ in order to show the variation in *shape* of the curves as a function of Rayleigh number and Prandtl number. It is evident from the figure that $W(z)$ and $\Theta(z)$ (when normalized in this manner) are remarkably insensitive to variations of R and P. This is consistent with the fact that in the case of 'free' boundaries, the shape of these functions is strictly independent of R and P.

After numerical determination of $W(z)$ and $\Theta(z)$, the heat flux was computed from (6.7) with R in the range $3 \leqslant \log_{10} R \leqslant 5$ and for $\log_{10} P = 0, \pm 1, \pm 2, \pm 3$. Figure 5 shows the resulting values of N, together with the corresponding A, σ and a. The diagrams on the left come from application of the uniqueness procedure (5.1) (maximum σ); those on the right come from application of (5.2) (maximum N). In general, the wave-numbers that maximize σ are substantially

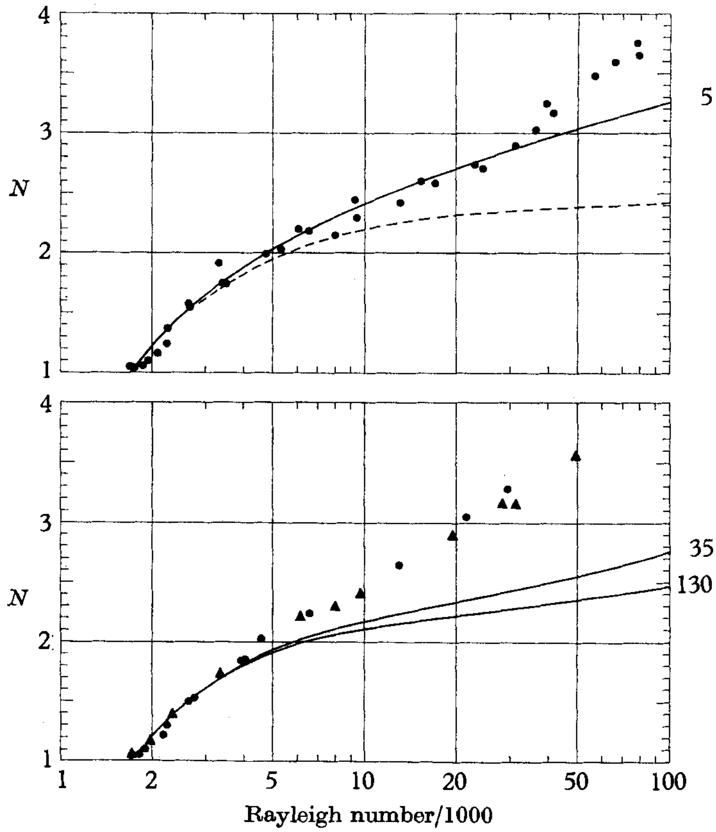

FIGURE 6. Experimental and theoretical determination of heat flux. Upper diagram: experimental data for water ($P \approx 5$) and theoretical curve for $P = 5$. (Broken curve is based upon Stuart's method, from (6.13).) Lower diagram: experimental data for silicone AK-3 (●) ($P \approx 35$) and for ethylene glycol (▲) ($P \approx 130$), and corresponding theoretical curves. Experimental data are from Silveston (1958). Theoretical results are based upon spectral truncation to one convective mode.

larger than those that maximize N, but these differences are not reflected significantly in σ or A, and especially not in N.

Chandrasekhar (1961) has summarized the principal experimental work on laboratory measurements of heat flux in the Bénard problem. For comparison with the theoretical results shown in figure 3, we quote the experimental work of Silveston (1958). Silveston's apparatus was one in which the fluid was contained in a cylinder of circular cross-section with rigid horizontal boundaries. The diameter of the cylinder was 198 mm. The distance between horizontal boundaries was variable: the minimum used was 1·45 mm, the maximum 12·98 mm. Heat flux was determined by an indirect method, in which heat losses involved in various parts of the apparatus were estimated and subtracted from the power supplied to the electrical heating element.

Figure 6 contains a representative selection of the Nusselt numbers tabulated by Silveston in the range $10^3 < R < 10^5$. The upper diagram shows data points for water, with Prandtl number in the range $3{\cdot}5 < P < 6{\cdot}6$. The solid curve is our theoretical spectral estimate on the basis of (6.7), with $P = 5$. The lower diagram shows data points for silicon oil AK-3 (circles) with Prandtl number in the range $35 < P < 36$, and for ethylene glycol (triangles) with Prandtl numbers in the range $126 < P < 138$. The theoretical spectral estimates for these data are shown by the solid curves $P = 35$ and $P = 130$. For water ($P = 5$) the agreement between computed and observed heat flux is very good out to the surprisingly high value $R = 35{,}000$. However, for higher Prandtl numbers, the spectral estimate is valid only to about $R = 4000$. The principal discrepancy between spectral estimates and experimental data in figure 6 is that the experimental data do not show the predicted inhibition of heat flux with increasing Prandtl number.

In a typical boundary-layer configuration of the horizontal-mean temperature $\tau(z)-Rz$, the transition between cold-boundary temperature $(-\frac{1}{2}R)$ and hot-boundary temperature $(+\frac{1}{2}R)$ takes place mainly in a thin conductive layer at each boundary, and the boundary layers are connected by a deep, quasi-isothermal convective layer. However, the spectral estimates of heat flux shown by the solid curves in figure 6 are based upon a truncation to only one convective mode, and therefore the corresponding horizontal-mean temperature cannot exhibit a boundary-layer configuration. Figure 7 shows the spectrally determined $R^{-1}\tau(z)-z$, which is the horizontal-mean temperature normalized to give the values $\pm\frac{1}{2}$ at the boundaries for all R. Only the 'cold' range $0 \leqslant z \leqslant \frac{1}{2}$ is shown, because this function is antisymmetric with respect to $z = 0$. (The function $\tau(z)$ was obtained from (6.4) by numerical quadrature.) At $R = 5000$, a temperature inversion appears midway between the boundaries. This is physically unrealistic, but can be regarded as consistent with the tendency for even a single mode to give a good spectral estimate of the heat flux (as shown in figure 6). In other words, with only a single mode present to govern the shape of the temperature profile, an inversion is inevitably the result of the large temperature gradients needed at the boundaries.

From his data, Silveston derived the following empirical power-law expressions for N:

laminar region	$(1700 < R < 3000)$: $N = 0{\cdot}0012R^{0{\cdot}90}$,
	$(4000 < R < 44{,}000)$: $N = 0{\cdot}24R^{0{\cdot}25}$;
transition region	$(R > 8000P^{0{\cdot}2})$: $N = 0{\cdot}30R^{0{\cdot}16}P^{0{\cdot}05}$;
turbulent region	$(R > 18{,}000P^{0{\cdot}2})$: $N = 0{\cdot}10R^{0{\cdot}31}P^{0{\cdot}05}$.

These relations conform to his data to within less than 10 %. Let

$$\Gamma \equiv (d\log N)/(d\log R) \tag{6.8}$$

denote the exponent of R in a power-law expression for N. In the first laminar region Silveston's empirical relation gives $\Gamma = 0{\cdot}90$. The value of Γ predicted by

(6.7) is a decreasing function of R. At $R = R_c$, where $\sigma_\epsilon = 0$ and $N = 1$, it is evident that (6.7) gives the marginal slope

$$\Gamma_c = R_c(d\sigma_\epsilon/dR)_c\,\bar{Q}_c/(\overline{Q_c - \bar{Q}_c)^2}, \tag{6.9}$$

which we now evaluate for comparison with the empirical value $\Gamma = 0{\cdot}90$.

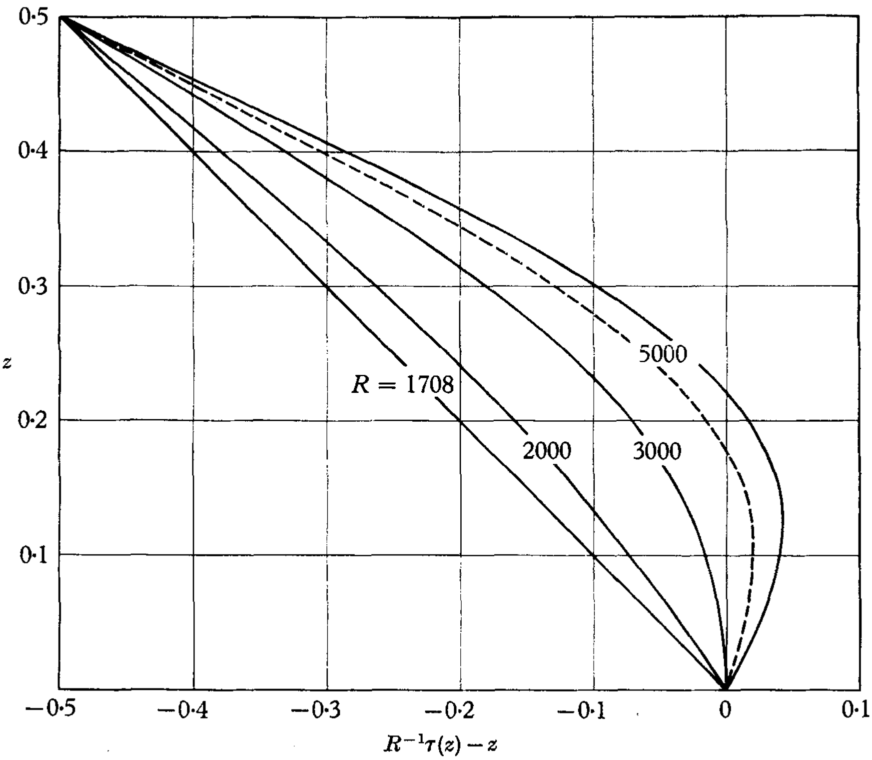

FIGURE 7. Normalized horizontal-mean temperature $R^{-1}\tau(z) - z$ (abscissa) in the range $0 \leqslant z \leqslant \frac{1}{2}$, based upon spectral truncation to one convective mode. Solid curves: $P = 1$ and $R = 1708$, 2000, 3000, 5000; broken curve: $P = 10$ and $R = 5000$.

To evaluate $(d\sigma_\epsilon/dR)_c$ we have first $(d\sigma_\epsilon/dR)_c = (\partial\sigma_\epsilon/\partial R)_c$, since $\partial\sigma_\epsilon/\partial a = 0$ at $R = R_c$. To evaluate $(\partial\sigma_\epsilon/\partial R)_c$, differentiate (3.3) with respect to R; with due regard for the dependence of $\mathscr{L}$ upon R, we find

$$\frac{\partial\sigma_\alpha}{\partial R}\mathbf{S}_\alpha = \begin{pmatrix}\mathbf{0}\\ w_\alpha\end{pmatrix} + \mathscr{L}\frac{\partial\mathbf{S}_\alpha}{\partial R} - \sigma_\alpha\frac{\partial\mathbf{S}_\alpha}{\partial R} - \frac{\partial\mathbf{G}_\alpha}{\partial R}.$$

Take the inner product of this equation with $\mathbf{S}_\alpha$; in view of the normalization (3.6) and the self-adjointness of $\mathscr{L}$

$$\frac{\partial\sigma_\alpha}{\partial R} = R^{-1}\overline{\overline{w_\alpha\theta_\alpha}} + \left\{\frac{\partial\mathbf{S}_\alpha}{\partial R}, \mathbf{G}_\alpha\right\} - \left\{\mathbf{S}_\alpha, \frac{\partial\mathbf{G}_\alpha}{\partial R}\right\}.$$

From the definition of $\mathbf{G}_\alpha$ (§3) we find that the second and third terms on the right here are zero, because $\mathbf{v}_\alpha$ and $\partial\mathbf{v}_\alpha/\partial R$ are solenoidal and have zero normal component on all boundaries. Hence the preceding equation reduces to

$$\partial\sigma_\alpha/\partial R = R^{-1}\overline{\overline{w_\alpha\theta_\alpha}}, \tag{6.10}$$

a useful identity. It follows now that

$$(d\sigma_\epsilon/dR)_c = (\partial\sigma_\epsilon/\partial R)_c = R_c^{-1}\bar{Q}_c,$$

where $Q_c \equiv \langle w_c \theta_c \rangle = W_c \Theta_c$.

If the preceding expression for $(d\sigma_\epsilon/dR)_c$ is used in (6.9), we get

$$\Gamma_c = \bar{Q}_c^2/\overline{(Q_c-\bar{Q}_c)^2} = [(\overline{Q_c^2}/\bar{Q}_c^2)-1]^{-1} \tag{6.11}$$

for the marginal slope of (6.7). In the case of 'free' boundaries the functions $W(z)$ and $\Theta(z)$ are proportional to $\cos \pi z$ (for all a, R, P), so Q_c is proportional to $\cos^2 \pi z$, and it is easy to verify from (6.11) that the marginal slope has the value (Malkus & Veronis 1958) $\Gamma_c = 2$. In the case of rigid horizontal boundaries the function Q_c was evaluated numerically as part of the process of computation of N from (6.7), and gave

$$\Gamma_c = 1{\cdot}4453 \tag{6.12}$$

from (6.11). This value exceeds by a factor of about 1·5 that deduced empirically by Silveston.

It is pertinent to note that the heat flux obtained by application of Stuart's (1958) method is

$$N = 1+\Gamma_c(R-R_c)/R, \tag{6.13}$$

where Γ_c is the expression given in (6.11). (This result is discussed in § 7.) The *marginal* slope of (6.13) evidently is exactly Γ_c, and thus is the same as that of (6.7)—although, as shown by the broken curve in the upper diagram of figure 6, (6.13) differs from (6.7) except in the neighbourhood of $R = R_c$. In connexion with (6.13), Malkus & Veronis (1958) estimated Γ_c as 1·51 on the basis of approximate analytic representations of $W_c(z)$, $\Theta_c(z)$ given by Pellew & Southwell (1940). On the other hand, Nakagawa (1960) found 0·8565 for Γ_c; this was deduced from exact analytic representations of $W_c(z)$, $\Theta_c(z)$ given by Reid & Harris (1958). The latter authors also tabulated numerical values of these functions for

$$z = 0{\cdot}00(0{\cdot}01)0{\cdot}50,$$

which were used by Chandrasekhar (1961, p. 614) to obtain (by numerical quadrature)

$$\bar{Q}_c = 2k \times 0{\cdot}229732, \quad \overline{Q_c^2} = 2k^2 \times 0{\cdot}178585,$$

the integrals needed for determination of Γ_c. (Here k is a factor the numerical value of which is irrelevant for Γ_c.) These yield exactly the value stated in (6.12), which therefore may be regarded as a confirmed value.†

The result given in (6.7) for a single primary mode can be extended to the truncation that retains the complete subspectrum of primary modes. Return to (4.10*b*). The last convection term is zero, as before, because $\mathbf{v}''' = 0$ in truncation to convective modes. The second and third convection terms in (4.10*b*) also are zero, as before, owing to the fact that θ and w are symmetric functions of z (since all primary modes are symmetric functions of z), and this is easily seen to give antisymmetric functions of z for all expressions whose volume integrals are

† Howard (1963) finds $\Gamma_c^{-1} = 0{\cdot}6919$ from Chandrasekhar's results; this agrees exactly with (6.12).

required in these terms. The calculation now proceeds in a straightforward manner and leads to the following results:

$$A_\alpha^2 = \kappa R \sum_\beta d_{\alpha\beta}\sigma_\beta, \tag{6.14}$$

$$D\tau = R \sum_\alpha \sum_\beta (Q_\alpha - \bar{Q}_\alpha)\, d_{\alpha\beta}\sigma_\beta, \tag{6.15}$$

$$N = 1 + \sum_\alpha \sum_\beta \bar{Q}_\alpha d_{\alpha\beta}, \tag{6.16}$$

where $Q_\alpha \equiv \langle w_\alpha \theta_\alpha \rangle = W_\alpha(z)\,\Theta_\alpha(z)$, and $d_{\alpha\beta}$ are the elements of the matrix inverse to the (real symmetric) matrix whose elements are $\overline{(Q_\alpha - \bar{Q}_\alpha)(Q_\beta - \bar{Q}_\beta)}$. It is easy to verify that (6.14), (6.15), (6.16) reduce to (6.3), (6.4), (6.7) when only a single element is retained in the subspectrum of primary modes.

7. The use of variance integrals

The heat-flux estimate (6.7) found by solution of truncated spectral-dynamic equations merits comparison with one that can be obtained by application of the general method developed by Stuart (1958) for estimating the amplitude of a secondary flow. Stuart's method as applied to the Bénard problem proceeds from the balance condition for thermal variance. To obtain a convenient form of this equation, multiply (2.1*b*) by $R^{-1}T$; after volume integration

$$\tfrac{1}{2}\,\partial\bar{\bar{H}}/\partial t = \overline{\overline{(\kappa\nabla^2 T + Rw)\,R^{-1}T}}.$$

Introduce the partition $T = \tau + \theta$, where $\tau = \langle T \rangle$ and $\theta = T - \langle T \rangle$, as explained previously. Since $\langle w \rangle = 0$ and $\langle \theta \rangle = 0$, we get

$$\tfrac{1}{2}\,\partial\bar{\bar{H}}/\partial t = \overline{\overline{(\kappa\nabla^2\theta + Rw)\,R^{-1}\theta}} - \kappa R^{-1}\overline{(D\tau)^2}, \tag{7.1}$$

after partial integration of the τ-term. In the steady state, this can be written

$$\overline{(\langle w\theta \rangle - \langle \overline{w\theta} \rangle)^2} = \kappa\overline{\overline{(\kappa\nabla^2\theta + Rw)\,\theta}}, \tag{7.2}$$

with the aid of (4.13) for $D\tau$.

In the procedure that led to (6.7) the convective part of the finite-amplitude state of convection was approximated in the form $A_\epsilon \mathbf{S}_\epsilon$, where $\mathbf{S}_\epsilon$ is a supercritical state of the fundamental primary mode and A_ϵ is the corresponding spectral amplitude. Following Stuart, we consider now an approximation $A\mathbf{S}_c$, where $\mathbf{S}_c$ is the *marginal* state of the primary mode and A is an amplitude to be determined. Thus, with $w = Aw_c$ and $\theta = A\theta_c$, we have from (7.2), after suppressing a factor A^2,

$$\overline{(Q_c - \bar{Q}_c)^2}\,A^2 = \kappa\overline{\overline{(\kappa\nabla^2\theta_c + Rw_c)\,\theta_c}},$$

where $Q_c \equiv \langle w_c \theta_c \rangle$. However, $\kappa\nabla^2\theta_c = -R_c w_c$ in view of (3.1*b*) and the fact that $\sigma_c = 0$, so the preceding equation after being solved for A^2 is (see Chandrasekhar 1961, p. 614)

$$A^2 = \kappa(R - R_c)\,\bar{Q}_c / \overline{(Q_c - \bar{Q}_c)^2},$$

in place of (6.3). Finally, with N as in (6.6), we obtain (6.13) in place of (6.7).

In comparing (6.13) and (6.7), it is necessary to emphasize that Stuart's approximation to a finite-amplitude state is not a spectral representation of the type employed in this study, because the convective mode is taken in a state (the marginal state) which corresponds to a Rayleigh number different from that associated with the supercritical state being approximated. The result obtained by this means therefore is not strictly comparable with that based upon a spectral approximation in terms of supercritical modal states. However, we have shown in § 6 that (6.7) and (6.13) have the same *marginal* slope (namely, Γ_c); in other words, the two results are equivalent in the limit $R \to R_c$.

We demonstrate now that Stuart's method can be formulated in such a way as to yield a result *identical* to (6.7). The first step must be to use the appropriate supercritical convective modal state $\mathbf{S}_\epsilon$ rather than the marginal state $\mathbf{S}_c$. However, this is not sufficient, for if the balance equation (7.1) for thermal variance is used, with $w = A_\epsilon w_\epsilon$ and $\theta = A_\epsilon \theta_\epsilon$, it is easy to see that the result for A_ϵ^2 is

$$A_\epsilon^2 = \kappa\sigma_\epsilon \overline{\Theta_\epsilon^2}/(\overline{Q_\epsilon - \bar{Q}_\epsilon})^2.$$

The ratio of this expression to (6.3) (the basis of (6.7)) is $R^{-1}\overline{\Theta_\epsilon^2}$, which in view of the normalization (3.18*b*) always differs from (in fact, is smaller than) unity.

To explain the discrepancy just noted, it is necessary to consider whether the balance condition for thermal variance is compatible with solutions of the *truncated* spectral-dynamic equations. In fact, generally it is not, because solutions of the truncated spectral equations, although compatible with the balance condition for kinetic *plus* thermal variance, are incompatible with the balance condition for kinetic variance, and therefore must be incompatible with the balance condition for thermal variance. This explains why (7.2) cannot be used as a starting point for a derivation of (6.7).†

To complete this discussion, we note that (6.7) can, in fact, be deduced from the balance condition for kinetic plus thermal variance. A partitioned form of the condition is needed for this purpose

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\overline{K+H}} = \sum_\alpha \sigma_\alpha''(A_\alpha'')^2 + \sum_\alpha \sigma_\alpha'''(A_\alpha''')^2 - \kappa R^{-1}\overline{(D\tau)^2}. \tag{7.3}$$

The truncation in question is one for which the first sum on the right is merely $\sigma_\epsilon A_\epsilon^2$, the second sum disappears, and $D\tau$ is given by (6.2). Equation (7.3) (in the steady state) then leads directly to (6.3) for A_ϵ^2, and hence to (6.7) for N.

8. Summary and conclusions

The governing hydrodynamical equations, based upon the Boussinesq approximations, are summarized in § 2. Variations of kinematic viscosity and thermal conductivity are ignored. Owing to the fact that the velocity is assumed solenoidal in the Boussinesq approximations, the pressure is a passive variable, and the state of convection is specified uniquely by a state vector consisting of the velocity and the dynamic part of the temperature. The inner product of two states is defined, and self-adjointness is established for the operator associated with the state vector in the governing equations.

† A more complete account of this and the next paragraph is given elsewhere (Platzman 1964, pp. 98–100).

In §3 the normal modes of the variance spectrum are introduced and their orthogonality is established. The modes are classified as thermal, convective, or kinetic. Thermal modes have no velocity field, whereas kinetic modes have no temperature field; both species are always damped. Convective modes are of two types: free convective modes, in which average convective flux of heat is from hot to cold boundary, and forced convective modes, in which heat flux is from cold to hot boundary. The latter are damped, the former may be amplified. The velocity field in a convective mode is purely poloidal if symmetry conditions are applied at lateral boundaries, as in tessellation of the infinite plane. In a kinetic mode the velocity field is purely toroidal.

The spectral-dynamic equations are formulated in §4. The solution of these equations gives the variance spectrum. Owing to the fact that the governing equations are bilinear in temperature, there can be no exchange of variance within the part of the variance spectrum that is spanned by thermal modes. For this reason it is possible to make an explicit elimination of the thermal part of the spectrum, and thus the spectral-dynamic equations can be closed on the convective and kinetic modes. Truncation of the spectral-dynamic equations is discussed. Although solutions of the truncated equations do not give approximation-in-the-mean to the variance spectrum, they may yield good approximations if the truncation is made judiciously.

In §5 two *ad hoc* procedures are introduced for establishing cell-scale uniqueness in steady tessellation of the infinite plane. The simpler from a numerical standpoint is maximization of the rate of amplification of the fundamental primary mode. Alternatively, the heat flux itself can be maximized. The latter procedure is in a sense more general, inasmuch as it is expressed in terms of an integral rather than a modal property of the convection. (Under the Boussinesq approximations, maximization of heat flux is equivalent to maximization of entropy production.) Appended to this section is a generalization of Christopherson's prototype formula for hexagonal tessellation.

Particular solutions of the truncated spectral-dynamic equations are obtained in §6. The case for which numerical results are given is one that excludes all kinetic modes and all convective modes except a single primary mode. (Primary modes are the 'lowest' modes with symmetric vertical structure.) Numerical estimates of heat flux are given on the basis of this truncation, for Rayleigh numbers up to 10^5 and Prandtl numbers in the range 10^{-3} to 10^3. These estimates are compared with the experimental data of Silveston. For water (Prandtl number 5), agreement is very satisfactory up to Rayleigh numbers of about 35,000. For higher Prandtl numbers agreement is good only to about 4000. In general, the principal discrepancy between the spectral estimate of heat flux and Silveston's experimental data, is that the latter do not show the predicted inhibition of heat flux with increasing Prandtl number.† The fact that the

† One of the original aims of this investigation was inclusion of the primary *forced* convective mode along with the corresponding free mode, in solution of the spectral equations and calculation of heat flux. There are no formal difficulties, but an awkward mathematical impediment intervened in the numerical normal-mode analysis of forced modes. Inclusion of a forced mode must reduce the heat flux still further, but the amount of this reduction is not yet known. I hope to be able to give these results in the near future.

spectral estimate for water is valid through transition to turbulence is surprising, especially in light of the circumstance that the thermal boundary layer cannot be represented adequately in highly supercritical states by a truncation that retains only the lowest vertical mode. This behaviour of the heat-flux estimate presumably must be ascribed to the fact that the truncated spectral-dynamic equations yield solutions compatible with the balance condition for kinetic plus thermal variance and thus satisfy an important integral constraint, no matter how the truncation is made.

The relation between the present work and that of Stuart is discussed in § 7. It is shown that Stuart's method can be formulated in such a way as to yield a result identical to that obtained through the spectral-dynamic equations when only a single convective mode is retained.

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